Factorial threefolds with $\mathbb{G}_a$-actions

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Abstract. The affine cancellation problem, which asks whether complex affine varieties with isomorphic cylinders are themselves isomorphic, has a positive solution for two dimensional varieties whose coordinate rings are unique factorization domains, in particular for $\mathbb{C}^2$, but counterexamples are found within normal surfaces (Danielewski surfaces) and factorial threefolds of logarithmic Kodaira dimension equal to 1. The latter are therefore remote from $\mathbb{C}^3$, the first unknown case where the base of one cylinder is an affine space. Locally trivial $\mathbb{G}_a$-actions play a significant role in these examples. Factorial threefolds admitting free $\mathbb{G}_a$-actions are discussed, especially a class of varieties with negative logarithmic Kodaira dimension which are total spaces of nonisomorphic $\mathbb{G}_a$-bundles. Some members of the class are shown to be isomorphic as abstract varieties, but it is unknown whether any members of the class constitute counterexamples to cancellation.

1. Introduction

Let $\mathbb{G}_a$ denote the additive group of complex numbers, and X a complex affine variety. Throughout, an action of $\mathbb{G}_a$ on X means an algebraic action. Every such action can be realized as the exponential of some locally nilpotent derivation δ of the coordinate ring $\mathbb{C}[X]$ of X and every locally nilpotent derivation gives rise to an action. The ring C_0 of $\mathbb{G}_a$-invariants in $\mathbb{C}[X]$ is equal to the ring of constants of the generating derivation.

An action $\sigma : \mathbb{G}_a \times X \to X$ is said to be equivariantly trivial if there is a variety Y for which X is $\mathbb{G}_a$-equivariantly isomorphic to $Y \times \mathbb{G}_a$, where $\mathbb{G}_a$ acts on $Y \times \mathbb{G}_a$ by translations on the second factor. In this case Y is affine and X the total space of a trivial $\mathbb{G}_a$-bundle over Y. A section for the bundle can be identified with the zero locus of a regular function $s \in \mathbb{C}[X]$ for which $\delta s = 1$. Such a function is called a slice; if one exists, $\mathbb{C}[X] = C_0[s]$ and $Y \cong$ Spec C_0. The action is said to be locally trivial if a geometric quotient $\pi : X \to Y$ exists in the category of algebraic spaces for which π is a principal $\mathbb{G}_a$-bundle. The action is said to be proper if the morphism

$$\sigma \times id : \mathbb{G}_a \times X \to X \times X$$

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is proper.

The main point of the paper is to investigate the class of hypersurfaces

$$X_{m,n} : x^m v - y^n u = 1$$

in $\mathbb{A}^4 = \mathrm{Spec}(\mathbb{C}[x, y, u, v])$, where m, n are positive integers. Since the polynomial $x^m v - y^n u = 1$ is an invariant for the $\mathbb{G}_a$ action

$$t(x, y, u, v) = (x, y, u + tx^m, v + ty^n)$$

on $\mathbb{A}^4$, the $X_{m,n}$ are stable under this action. Moreover, with $\mathbb{A}^2_* = \mathbb{A}^2 \setminus \{0\}$, the complement of the origin in the complex affine plane, the $X_{m,n}$, or rather the cohomology classes in $H^1(\mathbb{A}^2_*, O_{\mathbb{A}^2_*})$ of the Čech coycles $x^{-m}y^{-n} \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$, constitute a natural basis for the set of isomorphy classes of principal $\mathbb{G}_a$-bundles over $\mathbb{A}^2_*$. As seen below, these varieties can be viewed as globalizing the local structure of arbitrary smooth threefolds with locally trivial $\mathbb{G}_a$-action.

Similarly constructed threefolds were investigated in [**7**] as factorial counterexamples to a generalized cancellation problem:

$$\text{If } X \times \mathbb{A}^1 \cong Y \times \mathbb{A}^1 \text{ is } X \cong Y\ ?$$

Those examples arise from principal $\mathbb{G}_a$-bundles over the smooth locus of factorial surfaces of logarithmic Kodaira dimension 1, hence admit no nonconstant morphism from $\mathbb{A}^2$. As such they differ substantially from affine three space. As total spaces for $\mathbb{G}_a$-bundles over $\mathbb{A}^2_*$ the varieties $X_{m,n}$ considered here are closer in spirit to affine spaces and also have isomorphic cylinders. For distinct pairs (m, n) the corresponding varieties $X_{m,n}$ represent nonisomorphic $\mathbb{G}_a$-bundles, nevertheless some of them turn out to be isomorphic as varieties. We do not know as yet whether any pair of them provides a counterexample to cancellation.

For factorial affine varieties, i.e. those whose coordinate ring is a unique factorization domain (UFD), local triviality is equivalent with the intersection of C_0 with the image of δ generating the unit ideal in $\mathbb{C}[X]$ [**2**]. In this case the geometric quotient has the structure of a quasiaffine variety. To see this explicitly, consider a locally trivial $\mathbb{G}_a$-action on a factorial affine variety X and let $\delta(a_1), ..., \delta(a_n) \in C_0$ generate the unit ideal in $\mathbb{C}[X]$. The geometric quotient is embedded as an open subvariety in Spec R for an integrally closed ring R constructed as follows. Set $R_i = \mathbb{C}[X, \frac{1}{\delta(a_i)}]^{\mathbb{G}_a}$. Note that $\mathbb{C}[X, \frac{1}{\delta(a_i)}] = R_i[\frac{a_i}{\delta(a_i)}]$ so that R_i is a finitely generated $\mathbb{C}$-algebra, say

$$R_i = \mathbb{C}[b_{i1}, ..., b_{im}, \frac{1}{\delta(a_i)}],$$

with $b_{ij} \in C_0$. Define R to be the integral closure of $\mathbb{C}[b_{ij}, \delta(a_i) | 1 \leq i \leq n, 1 \leq j \leq m]$. The map $\pi : X \to \mathrm{Spec}(R)$ induced from the ring inclusion is smooth, hence open, and endows its image U with the structure of a geometric quotient. Moreover, for the open cover $\mathcal{U} = \{U_i = \mathrm{Spec}(R_i)\}_{1 \leq i \leq n}$

of U, the Čech cocycle

$$(\frac{a_1}{\delta(a_1)} - \frac{a_2}{\delta(a_2)}, \ldots, \frac{a_{n-1}}{\delta(a_{n-1})} - \frac{a_n}{\delta(a_n)}) \in C^1(\mathcal{U}, O_U)$$

defines the structure of $\pi : X \to U$ as a principal $\mathbb{G}_a$-bundle.

Recall that for an integral domain R with quotient field K, the transform of R with respect to the ideal I, written $T_I(R)$ is $\cup_{n>0}\{\alpha \in K | \alpha I^n \subset R\}$. The ring of $\mathbb{G}_a$-invariants is isomorphic to the ring of sections of the structure sheaf of Spec(R) on U, hence to $T_I(R)$ where I defines the closed subscheme $Z = \text{Spec}(R) \setminus U$ equipped with its reduced structure. It is clear from the construction that $I = \sqrt{(\delta(a_1), ..., \delta(a_n))}$. Since R is normal, $R = C_0$ if all components of Z have codimension ≥ 2 (equivalently if I is contained in no height one prime ideal of R).

In case the dimension of X is less than or equal to three, a theorem of Zariski [**14**, p.45] implies that C_0 is a finitely generated $\mathbb{C}$-algebra. Combined with the fact that the complement of a pure codimension one subvariety of a two dimensional normal affine variety is again an affine variety [**14**, p.45], the following ring theoretic criterion for triviality of a $\mathbb{G}_a$-bundle with a three dimensional affine factorial total space X is obtained:

Let X be an affine factorial threefold with a locally trivial $\mathbb{G}_a$-action generated by the locally nilpotent derivation δ of $C[X]$. With the ring R, ideal I,and Z, U as above, let $J = \cap_{\substack{I \subset \mathfrak{p} \\ ht\ \mathfrak{p}=1}} p$. Then $C_0 = T_J R$ and the action is equivariantly trivial if and only if $J = I$.

Indeed, since the complement W of the zero locus of J in Spec(R) is affine, every principal $\mathbb{G}_a$-bundle over W is trivial. If J properly contains I then U is equal to the complement of a finite but nonempty subset of Spec(C_0) (since I is a radical ideal, the Nullstellensatz implies that it has no embedded primes). In particular, U is not affine, and therefore neither is $U \times \mathbb{A}^1$.

More generally, a set-theoretically free action of an algebraic group on a normal variety admits a geometric quotient in the category of algebraic spaces, and the quotient map is a locally trivial fiber bundle in the étale topology. In this context, the properness of the action is then equivalent to the separatedness of the algebraic space quotient. Restricting now to proper $\mathbb{G}_a$-actions on normal quasiaffine threefolds, we obtain that the quotient exists as a quasiprojective variety and that the quotient map is locally trivial in the Zariski topology. Indeed, the quotient embeds as an open subset of a normal two dimensional analytic space. By Chow's lemma we know that a desingularization is quasiprojective, hence the quotient is the image of a quasiprojective variety under a proper morphism. As such it is again quasiprojective. That the quotient morphism is a Zariski locally trivial $\mathbb{G}_a$-bundle follows for instance from [**13**, Prop. 0.9]

All proper $\mathbb{G}_a$-actions on quasiaffine surfaces are known to be equivariantly trivial. For normal surfaces this was observed in [**3**], but again, an application of [**13**, Prop. 0.9] gives the result in general. All fixed point free actions on contractible threefolds are equivariantly trivial [**9**]. Examples of a nonlocally trivial proper action on $\mathbb{A}^5$ and on a smooth factorial fourfold are also discussed in [**3**]. The latter example is isomorphic to a cylinder over an affine variety which itself has the structure of the total space of a principal $\mathbb{G}_a$-bundle over the punctured affine plane $\mathbb{A}^2_*$, the topic of the subsequent sections of the present work. Except for special kinds of actions [**4**], the issue of local triviality of proper $\mathbb{G}_a$-actions on the affine four-space $\mathbb{A}^4$ is unsettled.

Replacing the smoothness hypothesis on the affine threefold X by factoriality and smoothness of the ring of $\mathbb{G}_a$-invariants, the following lemma of Miyanishi [**12**] leads to a local description of X.

Lemma 1. *Let $(\mathcal{O}, \mathfrak{m})$ be a regular local ring of dimension $n \geq 2$ and let A be a factorial, finitely generated $\mathcal{O}$-domain with $\mathcal{O} \hookrightarrow A$. Let $f : X \to Y$ be the morphism induced by the ring inclusion, where $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(\mathcal{O})$. Let $U = Y \setminus \{\mathfrak{m}\}$. Assume that $f_U : f^{-1}(U) \to U$ is an $\mathbb{A}^1$-bundle. Then either $X \cong Y \times \mathbb{A}^1$ or $f^{-1}(\mathfrak{m}) = \emptyset$ (the latter is only possible if $n = 2$).*

Corollary 1. *Let X be a smooth affine factorial threefold with a locally trivial $\mathbb{G}_a$-action that is not equivariantly trivial. Let δ be the derivation generating the action and $I = \sqrt{C_0 \cap \ker \delta \mathbb{C}[X]}$. Assume that C_0 is regular. For a maximal ideal $\mathfrak{m}$ of C_0, set $S = C_0 \setminus \mathfrak{m}$. Then with u, v denoting algebraically independent indeterminates,*

1. *$S^{-1}\mathbb{C}[X] \cong (C_0)_{\mathfrak{m}}[u]$ if $I \not\subseteq \mathfrak{m}$ and*
2. *$S^{-1}\mathbb{C}[X] \cong (C_0)_{\mathfrak{m}}[u, v]/(au - bv - 1)$ for some regular sequence (a, b) in $\mathfrak{m}$ if $I \subseteq \mathfrak{m}$.*

Proof. Since $S \subset C_0$, the action extends to $S^{-1}\mathbb{C}[X]$. Since the action is not equivariantly trivial , $IC_0 \neq C_0$. If $I \not\subseteq \mathfrak{m}$, then $S^{-1}\mathbb{C}[X]$ contains a slice. If $I \subseteq \mathfrak{m}$, then apply the lemma to $\mathcal{O} = (C_0)_{\mathfrak{m}}$ and $A = S^{-1}\mathbb{C}[X]$ to see that $\mathfrak{m}S^{-1}\mathbb{C}[X] = C[X]$. On the other hand, no proper principal ideal of $(C_0)_{\mathfrak{m}}$ blows up in $S^{-1}\mathbb{C}[X]$. □

The $X_{m,n}$ above represent a class of global versions of case 2.

Example 1. *Using a cover by the principal open subsets $x \neq 0, y \neq 0$, one sees that a "basis" for the principal $\mathbb{G}_a$-bundles over $\mathbb{A}^2_*$ is given by the affine varieties $X_{m,n} \subset \mathbb{A}^4$ defined by $x^m v - y^n u - 1 = 0$ where m, n are positive integers. These correspond to the Čech 1-cocycles $x^{-m}y^{-n}$ relative to the given cover of $\mathbb{A}^2_*$. The $X_{m,n}$ are smooth factorial threefolds with locally trivial $\mathbb{G}_a$-action generated by the locally nilpotent derivation $\delta : u \mapsto x^m \mapsto 0,\ \ v \mapsto y^n \mapsto 0$. Since $\mathbb{A}^2_*$ is a geometric quotient, $C_0 = \mathbb{C}[x, y] \cong \Gamma(\mathbb{A}^2_*, \mathcal{O}_{\mathbb{A}^2_*})$.*

Regularity of C_0 is not a necessary condition for the conclusion of the corollary.

EXAMPLE 2. *Let* $A = \mathbb{C}[x, y, z]/(x^2 + y^3 + z^5)$, *the coordinate ring for the analytically factorial singular surface* S, *and set* $U = S \setminus \{0\}$, *the smooth locus of* S . *Using the cover of* U *by the two open subsets* $S_x \cup S_y$,, *the cohomology classes of the Čech cocycles* $x^{-m}y^{-n}z^k$, $0 \le k \le 4$ *constitute a basis for* $H^1(U, O_U)$. *Consider the affine threefolds* $Z_{m,n,k} \subset \mathbb{A}^5$ *defined by*

$$x^m v - y^n u - z^k = 0,\ x^2 + y^3 + z^5 = 0, 0 \le k \le 4$$

for $0 \le k \le 4$. *With* $\pi_{m,n,k} : Z_{m,n,k} \to S$ *induced by the ring inclusion* $A \subset \mathbb{C}[Z_{m,n,k}]$, *the cocycle* $x^{-m}y^{-n}z^k$ *corresponds to the principal* $\mathbb{G}_a$-*bundle over* U *with total space* $E_{m,n,k} = \pi^{-1}(U)$ *and bundle projection* π. *Then* $E_{m,n,k}$ *is smooth, factorial, affine threefold admitting a locally trivial* $\mathbb{G}_a$-*action with quotient isomorphic to* U.

Smoothness of $E_{m,n,k}$ is clear from the defnition and factorialty follows from the fact that π is a Zariski fibration with base and fiber having trivial Picard group [**10**]. In fact $E_{m,n,k}$ is the quotient of a principal $\mathbb{G}_a$-bundle over $\mathbb{A}^2_*$ by the action of a finite group (the binary icosahedral group). We will see in the next section that all nontrivial $\mathbb{G}_a$-bundles over $\mathbb{A}^2_*$ have affine total spaces and deduce that the $E_{m,n,k}$ are affine by Chevalley's theorem.

The counterexamples to cancellation in [**7**] are constructed analogously to the $E_{m,n,k}$, i.e. as principal $\mathbb{G}_a$-bundles over surfaces

$$S_{a,b,c} : x^a + y^b + z^c = 0,\ \frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1$$

punctured at the singular point. The numerical criterion $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1$ enabled the distinction of the total spaces as affine varieties. We do not have analogous results for the $E_{m,n,k}$.

2. Principal $\mathbb{G}_a$-bundles over $\mathbb{A}^2_* = \mathbb{A}^2 \setminus \{0\}$

We start with the observation that the total spaces of nontrivial principal $\mathbb{G}_a$-bundles over $\mathbb{A}^2_*$ are always affine schemes. Indeed, a nontrivial Čech cocycle with value in $\mathcal{O}_{\mathbb{A}^2_*}$ for the covering of $\mathbb{A}^2_*$ by the open subsets $\{x \neq 0\}$ and $\{y \neq 0\}$ can be written as $g = p(x, y)x^{-m}y^{-n} \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ with $\deg_x p < m$, $\deg_y p < n$ and denote by $X(m, n, p)$ the total space for the so determined bundle over $\mathbb{A}^2_*$. Set

$$A = \mathbb{C}[x, y, u, v]/(x^n v - y^m u - p(x, y))$$

so that $X(m, n, p)$ is isomorphic to $\mathrm{Spec}(A) \setminus V(I)$ where $I = (x, y)A$.

If $p(0, 0) \neq 0$ then $X(m, n, p)$ is the zero locus of $x^m v - y^n u - p(x, y)$ in $\mathbb{A}^4$. Assume then that $p(0, 0) = 0$ and note that I is a height one prime ideal in A. We show that $IT_I(A) = T_I(A)$ to conclude that $X(m, n, p) = \mathrm{Spec}(A) \setminus V(I)$ is affine. Write $p(x, y) = \sum_{i=0}^{k} p_i(x)y^i$

Case 1: $p_0(x) \neq 0$. Let $a \geq 1$ be the multiplicity of 0 as a root of p and write $p_0(x) = x^a q_0(x)$. From

$$p(x,y) = x^a q_0(x) + y \sum_{i=1}^{k} p_i(x) y^{i-1}$$

we obtain

$$\frac{x^{m-a} - q_0}{y} = \frac{y^{n-1}u + \sum_{i=1}^{k} p_i(x) y^{i-1}}{x^a} \in T_I(A).$$

Thus $q_0(x) \in IT_I(A)$, but $q_0(0) \neq 0$ implies that $IT_I(A) = T_I(A)$.

Case 2: $p_0(x) = 0$. Write $p(x,y) = y^b \sum_{i=0}^{k-b} q_i(x) y^i$ with $q_0(x) \neq 0$. From

$$x^m v = y^b [y^{n-b} u + \sum_{i=0}^{k-b} q_i(x) y^i]$$

we obtain

$$w = \frac{v}{y^b} = \frac{y^{n-b}u + \sum_{i=0}^{k-b} q_i(x) y^i]}{x^m} \in T_I(A).$$

Then $v = y^b w$ so that in $T_I(A)$,

$$\begin{aligned} x^m y^b w - y^n u &= y^b \sum_{i=0}^{k-b} q_i(x) y^i \\ x^m w - y^{n-b} u &= \sum_{i=0}^{k-b} q_i(x) y^i. \end{aligned}$$

Replace A with $A' = \mathbb{C}[x,y,u,v]/(x^m w - y^{n-b}u - \sum_{i=0}^{k-b} q_i(x) y^i)$ to reduce to Case 1.

The topological universal cover of the $E_{m,n,k}$ of the previous section can be viewed as a principal $\mathbb{G}_a$-bundle over $\mathbb{A}^2_*$ and therefore as a "linear combination" of $X_{m,n}$ in the sense of Čech 1-cocycles. Indeed, for G the binary icosahedral group embedded in $GL_2(\mathbb{C})$ it is well known that $\mathbb{A}^2_*/G \cong U$ and that the étale mapping $\pi : \mathbb{A}^2_* \to \mathbb{A}^2_*/G$ is the universal topological covering. Thus, given a nontrivial $\mathbb{G}_a$-bundle $E_{m,n,k}$ over U, we obtain by base extension $\widetilde{E}_{m,n,k} = \mathbb{A}^2_* \times_U E_{m,n,k}$ a nontrivial principal $\mathbb{G}_a$-bundle over $\mathbb{A}^2_*$ whose total space yields the universal covering space of $E_{m,n,k}$. Moreover, $E_{m,n,k}$ is recovered as $\widetilde{E}_{m,n,k}/G$ where the finite group G acts freely on the first factor. It follows that each $\widetilde{E}_{m,n,k}$ is affine, smooth, and factorial. Finally, since $\widetilde{E}_{m,n,k} \to E_{m,n,k} \cong \widetilde{E}_{m,n,k}/G$ is a finite morphism, $E_{m,n,k}$ is affine (and smooth, factorial) by Chevalley's theorem.

The varieties $X_{m,n}$ (resp. $E_{m,n,k}$) in the preceding examples are total spaces for principal $\mathbb{G}_a$-bundles over the quasiaffine $\mathbb{A}^2_*$ (resp. U). The base extension $X_{m,n} \times_{\mathbb{A}^2_*} X_{p,q}$ is therefore a principal $\mathbb{G}_a$-bundle over both $X_{m,n}$

and $X_{p,q}$ for any m, n, p, q. Since they are affine, we obtain $X_{m,n} \times \mathbb{A}^1 \cong X_{p,q} \times \mathbb{A}^1$ and the same holds true for the cylinders over the $E_{m,n,k}$.

The well known Danielewski surfaces $D(n)$ given as the zero loci of the polynomials $y^2 - 2x^n z - 1$, $n > 0$ in $\mathbb{A}^3$ provide smooth but nonfactorial counterexamples to the generalized cancellation problem. In fact for $n \neq m$, $D(n)$ and $D(m)$ are not even homeomorphic in the Euclidean topology. In contrast, all $X_{m,n}$ (resp. $E_{m,n,k}$) are homeomorphic to $\mathbb{A}^2_* \times \mathbb{A}^1$ (resp. $U \times \mathbb{A}^1$ since, as principal bundles, the Euclidean topology of the base has a countable basis and the fiber is solid [**17**].

These considerations apply to any of the affine surfaces in $\mathbb{A}^3$ with a rational double point (quotient singularity) at the origin. Principal $\mathbb{G}_a$-bundles over the quasiaffine surface obtained by puncturing at the singular point will again produce potential counterexamples to the generalized affine cancellation problem. The surface defined by $x^2+y^3+z^5 = 0$ was chosen for its historical significance and the fact that its singularity is the unique one that is analytically factorial (hence "closest" to the affine plane $\mathbb{A}^2$). But we do not know whether or not the total spaces are isomorphic as varieties

On the other hand, the construction does provide cancellation counterexamples for other quotient singularities. The following example relies on the Makar Limanov invariant of an affine domain A :

$$ML(A) = \cap\{\ker\delta : \delta \text{ is a locally nilpotent derivation of } A\}$$

Example 3. *Let $U = S \setminus \{0\}$ where $S \subset \mathbb{A}^3$ is defined by*

$$x^2 + y^2 z + z^c = 0, \ c \geq 3.$$

It is well known that $S \cong \mathbb{A}^2/G$ where G is the binary dihedral group of order $4c$. Define $X_{m,n}(G) \subset S \times \mathbb{A}^2$ by $x^m v - y^n u - 1 = 0$ where m, n are positive integers. Then

(1) $X_{m,n}(G)$ is not factorial, but $ML(\mathbb{C}[X_{m,n}(G)]) = \mathbb{C}[S]$. To see this, let δ be locally nilpotent with kernel R, and $K = \mathrm{Frac}(R)$. Then the extension of δ to $K \otimes_R \mathbb{C}[X_{m,n}(G)]$ has a slice s and

$$K \otimes_R \mathbb{C}[X_{m,n}(G)] \cong K[s], \ \delta(s) = 1$$

and K is the kernel of the extended derivation. If $\deg_s(z) > 0$, then $z(s)$ divides both $x(s), y(s)$ contradicting $x^m v - y^n u = 1$. Thus $z \in R$ from which it follows that both $x, y \in R$.

(2) $X_{m,n}(G) \cong X_{p,q}(G)$ if and only if $(m,n) = (p,q)$. This follows from arguments analogous to those in [**7**] and the fact that any automorphism of $\mathbb{C}[Y]$ is of the form $(x,y,z) \mapsto (\mu_1 x, \mu_2 y, \mu_3 z)$ for certain roots of unity μ_i [**11**].

(3) $\mathbb{A}^2_* \times_U X_{m,n}(G) \to X_{m,n}(G)$ is an étale covering.

Note that the total spaces $\mathbb{A}^2_* \times_U X_{m,n}(G)$ of these principal $\mathbb{G}_a$-bundles over $\mathbb{A}^2_*$ are not isomorphic as G-varieties, but it is possible that there is some abstract isomorphism between them.

It is a classical result going back to Klein that the ring of invariants $\mathbb{C}[x,y]^G$ for the natural action of a finite subgroup G of $SL_2(\mathbb{C})$ on $\mathbb{A}^2$ corresponding to the quotient surface singularities can be generated by 3 homogeneous polynomials. As a consequence, affine threefolds of the type $X_{m,n}(G)$ for such G can be interpreted as quasihomogeneous varieties with respect to a grading determined by the generators of $\mathbb{C}[x,y]^G$. Properties of such quasihomogeneous threefolds will be addressed in future work.

3. Isomorphy Classes of $X_{m,n}$

Recall the hypersurfaces $X_{m,n} \subset \mathbb{A}^4 = \mathrm{Spec}(\mathbb{C}[x,y,u,v])$ defined by

$$x^m v - y^n u = 1$$

with $mn \neq 0$. The main result of this section shows that at least for some distinct pairs $\{m,n\} \neq \{p,q\}$ we have $X_{m,n} \cong X_{p,q}$.

In his investigation of complex surfaces with $\mathbb{G}_a$-actions [**5**], tom Dieck observed that $X_{m,m}$ admits a free action of a semidirect product structure G_{2m} on $\mathbb{G}_m \times \mathbb{G}_a$ for which $X_{m,m}/G_{2m} \cong \mathbb{P}^1$ and $X_{m,m} \to \mathbb{P}^1$ is a principal G_{2m}-bundle. Moreover, the induced mapping $X_{m,m}/\mathbb{G}_m \to \mathbb{P}^1$ is diffeomorphic to the complex line bundle $\mathcal{O}_{\mathbb{P}^1}(-2m)$. The quotient $X_{m,m}/\mathbb{G}_m$ turns out to be isomorphic to the Danielewski surface $D(m) : y^2 - 2x^m z = 1$ and the latter can thus be distinguished topologically for distinct values of m. A related construction enables us to prove the existence of an algebraic isomorphism $X_{m,n} \cong X_{p,q}$ if $m+n = p+q$ albeit with a different analysis of the structure.

For a given (m,n) we have actions of $\mathbb{G}_m$ and $\mathbb{G}_a$ on $X_{m,n}$ given respectively by

$$\begin{array}{ccc} \mathbb{G}_m \times X_{m,n} & \to & X_{m,n} \\ (\lambda,(x,y,u,v)) & \longmapsto & (\lambda x, \lambda y, \lambda^{-n} u, \lambda^{-m} v). \end{array}$$

and

$$\begin{array}{ccc} \mathbb{G}_a \times X_{m,n} & \to & X_{m,n} \\ (t,(x,y,u,v)) & \longmapsto & (x, y, u + t x^m, v + t y^n) \end{array}$$

The $\mathbb{G}_a$-action is generated by the derivation $\delta : u \mapsto x^m \mapsto 0,\ v \mapsto y^n \mapsto 0$ and the so determined principal $\mathbb{G}_a$-bundle $\pi : X_{m,n} \to \mathbb{A}^2_*$ is trivialized over the open covering of $\mathbb{A}^2_*$ by the open subsets

$$U_x = \{x \neq 0\} \text{ and } U_y = \{y \neq 0\}$$

Writing $\theta : (\mathbb{G}_m \times \mathbb{G}_a) \times X_{m,n} \to X_{m,n}$ for the composite action, note that

$$\begin{aligned} \theta((\lambda,t),(x,y,u,v)) &= (\lambda x, \lambda y, \lambda^{-n} u + t\lambda^{-n} x^m, \lambda^{-m} v + t\lambda^{-m} y^n) \\ &= (\lambda x, \lambda y, \lambda^{-n} u + t\lambda^{-(m+n)} (\lambda x)^m, \lambda^{-m} v + t\lambda^{-(m+n)} (\lambda y)^n) \\ &= \theta((\lambda, \lambda^{-(m+n)} t),(x,y,u,v)) \end{aligned}$$

Set $d = m+n$. From the last equation it is clear that $X_{m,n}$ is equipped with a free action of the semidirect product $G_d = \mathbb{G}_m \ltimes_d \mathbb{G}_a$ with multiplication

$$(\lambda, t)\ (\lambda', t') = (\lambda\lambda', t + \lambda^d t').$$

The open covering $\pi^{-1}(U_x) \cup \pi^{-1}(U_x)$ that trivializes the $\mathbb{G}_a$-action on $X_{m,n}$ is clearly G_d-stable as well. We let

$$\begin{array}{rcl} V_x &=& \pi^{-1}(U_x) \cong \operatorname{Spec}(\mathbb{C}[x^{\pm 1}, y, u, v]/(x^m v - y^n u - 1)) \cong \operatorname{Spec}(\mathbb{C}[x^{\pm 1}, y, u]) \\ V_y &=& \pi^{-1}(U_y) \cong \operatorname{Spec}(\mathbb{C}[x, y^{\pm 1}, u, v]/(x^m v - y^n u - 1)) \cong \operatorname{Spec}(\mathbb{C}[x, y^{\pm 1}, v]). \end{array}$$

Setting

$$(u_1, T_1, L_1) = (yx^{-1}, x^n u, x) \text{ and } (u_2, T_2, L_2) = (xy^{-1}, y^m v, y)$$

and observing that u_i are G_d-invariant, T_i are $\mathbb{G}_m$-invariant and translated by the $\mathbb{G}_a$-action (vice versa for L_i), we obtain G_d-equivariant trivializations

$$\begin{array}{rcl} V_x &\cong& \operatorname{Spec}(\mathbb{C}[x^{\pm 1}, y, u]) \cong \operatorname{Spec}(\mathbb{C}[u_1][T_1, L_1, L_1^{-1}]) \cong \mathbb{A}^1 \times G_d \\ V_y &\cong& \operatorname{Spec}(\mathbb{C}[x, y^{\pm 1}, v\ ]) \cong \operatorname{Spec}(\mathbb{C}[u_2][T_2, L_2, L_2^{-1}]) \cong \mathbb{A}^1 \times G_d. \end{array}$$

Gluing $V_x/G_d \cong \operatorname{Spec}(\mathbb{C}[u_1])$ and $V_y/G_d \cong \operatorname{Spec}(\mathbb{C}[u_2])$ over their intersection $\operatorname{Spec}(\mathbb{C}[u_1, u_1^{-1}])$ via $u_1 \mapsto u_2^{-1}$ we obtain a quotient

$$\pi_{m,n} : X_{m,n} \to C_{m,n} \cong \mathbb{P}^1$$

which is a Zariski locally trivial principal G_d-bundle. Transition isomorphisms are given by $L_2 = y = u_1 L_1$ and

$$\begin{array}{rcl} T_2 &=& y^m v = (\dfrac{y}{x})^m x^m v = u_1^m (1 + y^n u) = u_1^m (1 + u_1^n T_1) \\ &=& u_1^m + u_1^d T_1 \end{array}$$

proving

Proposition 1. *The threefold $X_{m,n}$ admits the structure of a principal G_{m+n}-bundle over $\mathbb{P}^1$ that is locally trivial for the Zariski topology. The transition isomorphisms for this bundle are*

$$(L, T) \longmapsto (uL, u^{m+n} T + u^m).$$

In contrast to the differential-topological context of [**5**] we cannot conclude that the induced mapping $X_{m,n}/\mathbb{G}_m \to C_{m,n} \cong \mathbb{P}^1$ is *algebraically* isomorphic to the line bundle $O_{\mathbb{P}^1}(-(m+n))$ (the translation part of the transition isomorphism cannot be algebraically contracted away). Nevertheless, we can conclude from the discussion above with $d = m+n$, that

Corollary 2. (1) *The quotient $X_{m,n}/\mathbb{G}_m$ is isomorphic to the affine surface $S_{d,m}$ which is the total space of the $\mathcal{O}_{\mathbb{P}^1}(-d)$-torsor $\rho : S_{d,m} \to \mathbb{P}^1$ with transition isomorphism $T \mapsto u^d T + u^m$.*

(2) $\operatorname{Pic}(S_{d,m}) \cong \mathrm{H}^1(S_{d,m}, \mathcal{O}^*_{S_{d,m}}) \cong \mathbb{Z}$ *and the isomorphism class of the $\mathbb{G}_m$-bundle $X_{m,n} \to S_{d,m}$ is a generator of* $\operatorname{Pic}(S_{d,m})$.

Proof. The second assertion follows from the fact that every $\mathbb{A}^1$-bundle over $\mathbb{P}^1$ can be realized as the complement of a section C in a suitable $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ [**18**, Theorem 2.3] ($S_{d,m}$ is explicitly realized this way in the proof of Theorem 1 below). The latter have divisor class group isomorphic to $\mathbb{Z}^2$, generated by the classes of a fiber and a section that can be chosen to be precisely C. Thus the divisor class group of $S_{d,m}$ is isomorphic to $\mathbb{Z}$, generated for instance by the class of a fiber of ρ. $\square$

The isomorphism class of $X_{m,n}$ as a variety is therefore determined by a generator of $\mathrm{Pic}(S_{d,m})$, and moreover is independent of the choice of generator. Indeed, choosing the other generator yields the $\mathbb{G}_m$-bundle $\tilde{X}_{m,n}$ over $S_{d,m}$ with transition isomorphism $(L,T) \longmapsto (u^{-1}L, u^dT + u^m)$. The coordinate change $\tilde{L} = L^{-1}$ on the fibers of the bundle yields an isomorphism $\tilde{X}_{m,n} \xrightarrow{\sim} X_{m,n}$.

The structure of $S_{d,m}$ as an $\mathcal{O}_{\mathbb{P}^1}(-d)$-torsor (actually as an $\mathbb{A}^1$-bundle over $\mathbb{P}^1$) enables the application of a famous theorem of Danilov-Gizatullin [**1**] to determine its isomorphy class as an affine surface (for a recent proof of this theorem see [**6**]).

The transition isomorphism $T \longmapsto u^dT + u^m$ determines an $\mathbb{A}^1$-bundle over $\mathbb{P}^1$ which in turn can be completed to the $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{E})$, where $\mathcal{E}$ is the rank 2 vector bundle over $\mathbb{P}^1$ with transition matrix

$$M = \begin{bmatrix} u^d & u^m \\ 0 & 1 \end{bmatrix}$$

corresponding to a non trivial extension

$$0 \to O_{\mathbb{P}^1}(-d) \to \mathcal{E} \to O_{\mathbb{P}^1} \to 0.$$

Since the goal is to show that $X_{m,n} \cong X_{p,q}$ if $p+q = m+n$, and obviously $X_{m,n} \cong X_{n,m}$, we will assume that $m \geq n$. The results and terminology about ruled surfaces in what follows can be found in [**8**, V.2].

Lemma 2. *The total space of the $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{E}) \to \mathbb{P}^1$ is isomorphic to the Nagata-Hirzebruch surface $\mathbb{F}_{2m-d}$.*

Proof. It suffices to show that $\mathcal{E}\otimes\mathcal{O}_{\mathbb{P}^1}(m)$ is normalized, i.e. that

$$\mathrm{H}^0(\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^1}(m)) \neq 0 \text{ and } \mathrm{H}^0(\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^1}(m-1)) = 0.$$

Indeed, if these conditions hold, then by counting degrees, $\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^1}(m) \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2m-d)$ and, as is well known, $\mathbb{P}(\mathcal{E}) \cong \mathbb{P}(\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^1}(m)) \cong \mathbb{F}_{2m-d}$.

The vector bundle $\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^1}(j)$ has transition matrix

$$M_j = \begin{bmatrix} u^{d-j} & u^{m-j} \\ 0 & u^{-j} \end{bmatrix}$$

and a nonzero global section for this bundle corresponds to a pair of nonzero vectors

$$g = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \in \mathbb{C}[u] \times \mathbb{C}[u] \quad \text{and} \quad h = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \in \mathbb{C}[u^{-1}] \times \mathbb{C}[u^{-1}]$$

with $M_j \cdot g = h$ For $j = m-1$ this would mean

$$\begin{array}{rcl} h_1 & = & u^{d-m+1}g_1 + ug_2 \\ h_2 & = & u(u^{d-m}g_1 + g_2) \end{array}$$

which is impossible as $d > m$. For $j = m$ we have, for example,

$$g = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad h = \begin{bmatrix} 1 \\ u^{-m} \end{bmatrix}.$$

$\square$

THEOREM 1. *Let $d = p + q = m + n$. Then $X_{m,n} \cong X_{p,q}$ as abstract varieties.*

PROOF. As noted above, since $X_{m,n} \cong X_{n,m}$, we may assume without loss of generality that $m \geq n$, so that $2m \geq d$. With the notation of the corollary, $X_{m,n}$ is the total space of a $\mathbb{G}_m$-bundle over $S_{d,m}$ which in turn is the total space of an $O_{\mathbb{P}^1}(-d)$-torsor over $\mathbb{P}^1$. Choosing homogeneous coordinates $[u : v]$ on the fibers F of the $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{E}) \cong \mathbb{F}_{2m-d} \to \mathbb{P}^1$, $S_{d,m}$ is obtained as the complement of the section $v = 0$. The associated divisor is $C + mF$ where C is the special section with self-intersection $d - 2m$. The self-intersection of $C + mF$ is calculated as

$$(C + mF)^2 = C^2 + 2mC.F = -(2m - d) + 2m = d$$

and thus depends only on the sum $d = p + q = m + n$.

The theorem of Danilov-Gizatullin says exactly that $S_{d,m}$ and $S_{d,p}$ are isomorphic as affine surfaces. An isomorphism $S_{d,m} \xrightarrow{\phi} S_{d,p}$ carries a generator of $\mathrm{Pic}(S_{d,p})$ to a generator of $\mathrm{Pic}(S_{d,m})$ under the induced isomorphism

$$\mathrm{Pic}(S_{d,p}) \xrightarrow{\phi^*} \mathrm{Pic}(S_{d,m}).$$

Since the class of $X_{m,n}$ (and $\tilde{X}_{m,n}$) generates $\mathrm{Pic}(S_{d,m})$ and $X_{m,n} \cong \tilde{X}_{m,n}$ we obtain the isomorphism $X_{m,n} \cong X_{p,q}$. $\square$

While it appears to be rather difficult to construct an explicit isomorphism between even the first interesting examples, $X_{2,2}$ and $X_{3,1}$, one can check that $X_{2,2}$ admits the structure of the principal $\mathbb{G}_a$-bundle over $\mathbb{A}^2_*$ corresponding to the Čech cocycle $x^{-3}y$.

EXAMPLE 4. *Set*

$$\begin{array}{rcl} a(x,y,u,v) & = & x - \dfrac{1}{2}y \\ b(x,y,u,v) & = & \dfrac{(6x-y)}{8}v - \dfrac{(3y-2x)}{2}u \\ w(x,y,u,v) & = & \dfrac{5}{16}v^2x + \dfrac{5}{2}vxu - \dfrac{1}{32}v^2y - \dfrac{5}{4}vyu + u^2x - \dfrac{5}{2}u^2y \end{array}$$

The morphism

$$\begin{aligned}\tilde{\pi} \quad : \quad & X_{2,2} \to \mathbb{A}^2_* \cong \operatorname{Spec}(\mathbb{C}[a,b]) \setminus \{0\}. \\ & (x,y,u,v) \mapsto (a(x,y,u,v), b(x,y,u,v))\end{aligned}$$

is a $\mathbb{G}_a$-bundle corresponding to the Čech cocycle $a^{-3}b$. In particular, one can check that the assignment

$$x \mapsto \frac{a^3}{3},\ y \mapsto a^3,\ u \mapsto xb - \frac{1}{4},\ v \mapsto 2yb - 1$$

extends to a locally nilpotent $\mathbb{C}$-derivation δ on $A_{2,2} = \mathbb{C}[x,y,u,v]/(x^2v - y^2u - 1)$ with $a, b \in \ker(\delta)$ and

$$\delta(y + a + ab) = a^3,\ \delta(w) = b.$$

Since $(a^3, b)A_{2,2} = A_{2,2}$, we obtain a locally trivial $\mathbb{G}_a$-action on $X_{2,2}$ with quotient isomorphic to $\mathbb{A}^2_*$ corresponding to the Čech cocycle

$$\frac{y + a + ab}{a^3} - \frac{w}{b} = \frac{1}{a^3 b}.$$

4. A more general class of examples

A refinement of the arguments in the previous section enables a description of the isomorphy classes of total spaces of principal $\mathbb{G}_a$-bundles defined by arbitrary homogeneous bivariate polynomials in terms of the $X_{m,n}$. Let $f, g \in \mathbb{C}[x,y]$ be arbitrary homogeneous polynomials such that $\operatorname{Spec}(\mathbb{C}[x,y]/(f,g))$ is supported at the origin of $\mathbb{A}^2$ and consider affine threefolds $X_{f,g}$ in $\mathbb{A}^4 = \operatorname{Spec}(\mathbb{C}[x,y,u,v])$ defined by the equations $fv - gu - 1 = 0$. Again, these threefolds come naturally equiped with the structure $\pi_{f,g} : X_{f,g} \to \mathbb{A}^2_*$ of $\mathbb{G}_a$-bundles over the complement of the origin in $\mathbb{A}^2 = \operatorname{Spec}(\mathbb{C}[x,y])$ by restricting the projection $\mathrm{p}_{x,y} : \mathbb{A}^4 \to \mathbb{A}^2$ to $X_{f,g}$. The latter becomes trivial on the covering of $\mathbb{A}^2_*$ by means of the principal affine open subsets

$$U_f = \operatorname{Spec}\left(\mathbb{C}[x,y]_f\right) \text{ and } U_g = \operatorname{Spec}\left(\mathbb{C}[x,y]_g\right).$$

Suppose that f and g are homogeneous, say of degree m and n respectively. Similar to the case of the $X_{m,n}$ of the previous section, the $\mathbb{G}_m$-action $\lambda \cdot (x,y) = (\lambda x, \lambda y)$ on $\mathbb{A}^2$ lifts to a free $\mathbb{G}_m$ -action on $X_{f,g}$ defined by

$$\lambda \cdot (x,y,u,v) = \left(\lambda x, \lambda y, \lambda^{-m} u, \lambda^{-n} v\right)$$

which is compatible with the structural map $\pi_{f,g} : X_{f,g} \to \mathbb{A}^2_*$. It follows that $\pi_{f,g}$ descends to an $\mathbb{A}^1$-bundle $\rho_{f,g} : X_{f,g}/\mathbb{G}_m \to \mathbb{A}^2_*/\mathbb{G}_m \simeq \mathbb{P}^1$. The latter is an $\mathcal{O}_{\mathbb{P}^1}(-n-m)$-torsor which can be described more explicitly as follows. Recall that for every pair of integers $m, n \geq 1$, the quotient of $\mathbb{A}^2_* \times \mathbb{A}^2_*$ by the free $\mathbb{G}_m^2$-action given by

$$(\lambda, \mu) \cdot (x,y,u,v) = \left(\lambda x, \lambda y, \mu\lambda^{-n} u, \mu\lambda^{-m} v\right)$$

is the surface scroll

$$\pi : \mathbb{F}(m,n) = \mathbb{P}\left(\mathcal{O}_{\mathbb{P}^1}(m) \oplus \mathcal{O}_{\mathbb{P}^1}(n)\right) \to \mathbb{P}^1,$$

with structural morphism π induced by the $\mathbb{G}_m^2$-invariant morphism $\mathbb{A}_*^2 \times \mathbb{A}_*^2 \to \mathbb{P}^1$, $(x,y,u,v) \mapsto [x:y]$. Clearly, $X_{f,g}$ is contained in the complement of the $\mathbb{G}_m^2$-invariant closed subset D of $\mathbb{A}_*^2 \times \mathbb{A}_*^2$ with equation $fv - gu = 0$. Furthermore, $X_{f,g}$ is invariant under the action of the first factor of $\mathbb{G}_m^2$ and is a slice for the action of the second factor of $\mathbb{G}_m^2$ of $\mathbb{A}_*^2 \times \mathbb{A}_*^2 \setminus D$. Thus the quotient map restricts to a $\mathbb{G}_m = \mathbb{G}_m \times \{1\}$-bundle

$$\rho : X_{f,g} \to X_{f,g}/\mathbb{G}_m \times \{1\} \simeq \left(\mathbb{A}_*^2 \times \mathbb{A}_*^2 \setminus D\right)/\mathbb{G}_m^2 = \mathbb{F}(m,n) \setminus \Delta_{f,g}.$$

over the complement of the image $\Delta_{f,g}$ of D in $\mathbb{F}(m-n)$.

The self-intersection of the curve $\Delta_{f,g}$, which is a section of π, can determined as follows. The images by the quotient map of the $\mathbb{G}_m^2$-invariant subsets $\{v=0\}$ and $\{u=0\}$ define two sections C_v and C_u of π such that $C_v \cdot C_u = 0$. Letting L be a fiber of π, the divisors $mL + C_v$ and $nL + C_u$ are linearly equivalent since they differ by the divisor of the rational function $h = fv/gu$ on $\mathbb{F}(m,n)$. Since $\Delta_{f,g}$ belongs to the complete linear system generated by these divisors, we conclude that

$$\Delta_{f,g}^2 = (mL + C_v) \cdot (nL + C_u) = m + n.$$

The above description implies that $X_{f,g}/\mathbb{G}_m$ is isomorphic to the complement of a section with self-intersection $m+n$ in $\mathbb{F}(m,n) \simeq \mathbb{F}_{|m-n|}$, and so, the $\mathbb{P}^1$-bundle $\mathbb{F}_{|m-n|} \to \mathbb{P}^1$ restricts to an $\mathcal{O}_{\mathbb{P}^1}(-m-n)$-torsor

$$\rho_{f,g} : X_{f,g}/\mathbb{G}_m \to \mathbb{A}_*^2/\mathbb{G}_m \simeq \mathbb{P}^1.$$

Combined with the results of the previous section, this the leads to the following

Proposition 2. *With the notation above, $X_{f,g}$ is isomorphic to $X_{m,n}$.*

Proof. Recall that $X_{m,n}$ has the structure of a $\mathbb{G}_m$-bundle over an affine surface $S_{d,m}$ obtained as the complement of certain section $\Delta_{m,n}$ with self-intersection $m+n$ in $\mathbb{F}_{|m-n|}$. By virtue of Gizatullin's classification [**1**], $\mathbb{F}_{|m-n|} \setminus \Delta_{f,g} \simeq \mathbb{F}_{|m-n|} \setminus \Delta_{m,n} = S_{d,m}$ as abstract affine surfaces. It follows that $Z = X_{m,n} \times_{S_{d,m}} X_{f,g}$ is a $\mathbb{G}_m$-bundle over both $X_{m,n}$ and $X_{f,g}$ via the first and the second projection respectively. Since the Picard groups of $X_{m,n}$ and $X_{f,g}$ are both trivial, these $\mathbb{G}_m$-bundles are actually trivial, which yields an isomorphism $X_{m,n} \times \mathbb{A}_*^1 \simeq X_{f,g} \times \mathbb{A}_*^1$ as affine varieties. Since $X_{m,n}$ and $X_{f,g}$ are both irreducible and have no invertible functions except nonzero constants, the latter descends to an isomorphism $X_{m,n} \simeq X_{f,g}$. □

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